Hilbert space $(\mathcal{H}_n, +, \cdot, \langle\cdot,\cdot\rangle)$

- $\mathcal{H}_n = \{a_0|0\rangle + a_1|1\rangle + \cdots + a_{n-1}|n-1\rangle \mid a_i \in \mathbb{C}\}$, $\quad \mathcal{B} = \{|0\rangle, |1\rangle, \cdots, |n-1\rangle\}$ orthonormal basis

$$\mathbb{C}^n = \left\{ \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} \;\middle|\; " \right\} \qquad = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

$n = 3$

- $\mathcal{H}_3 = \{a|0\rangle + b|1\rangle + c|2\rangle \mid a, b, c \in \mathbb{C}\}$ $\quad \mathcal{B} = \{|0\rangle, |1\rangle, |2\rangle\}$

$$\mathbb{C}_3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \;\middle|\; " \right\} \qquad = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- $\mathcal{H}_3 \ni |u\rangle = a|0\rangle + b|1\rangle + c|2\rangle = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = u \in \mathbb{C}^3$

  $|v\rangle = x|0\rangle + y|1\rangle + z|2\rangle = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = v$

- (1) + (2) $\quad 3|u\rangle + 2|v\rangle = \begin{bmatrix} 3a + 2x \\ 3b + 2y \\ 3c + 2z \end{bmatrix} = 3u + 2v$

- $\hat{\mathcal{H}}_3 = L(\mathcal{H}_3, \mathbb{C}) \ni \hat{u} = \widehat{|u\rangle} = [\bar{a}\ \bar{b}\ \bar{c}] : \mathcal{H}_3 \longrightarrow \mathbb{C}$ (ket)

  $\hat{\mathcal{H}}_3 = \mathcal{H}_3^*$, $\quad \hat{u} = u^*$, $\quad \widehat{|u\rangle} = \langle u|$ bra

  $v \longmapsto u^* v = [\bar{a}\ \bar{b}\ \bar{c}] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \bar{a}x + \bar{b}y + \bar{c}z$

$\mathcal{B}^* = \{ e_1^* = [1\ 0\ 0], e_2^* = [0\ 1\ 0], e_3^* = [0\ 0\ 1] \}$ $\quad e_i^* e_j = \delta_{ij}$

$= \{ \langle 0|, \langle 1|, \langle 2| \}$, $\quad = \langle i|j\rangle$

- (3) $\langle u, v\rangle = u^* v = [\bar{a}\ \bar{b}\ \bar{c}] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \bar{a}x + \bar{b}y + \bar{c}z$ (内積)

  $\langle u, v\rangle = \langle |u\rangle, |v\rangle\rangle = \langle u||v\rangle \overset{def}{=} \langle u|v\rangle$

- (4) $|u\rangle\langle v| = uv^* = \begin{bmatrix} a \\ b \\ c \end{bmatrix} [\bar{x}\ \bar{y}\ \bar{z}] = \begin{bmatrix} a\bar{x} & a\bar{y} & a\bar{z} \\ b\bar{x} & b\bar{y} & b\bar{z} \\ c\bar{x} & c\bar{y} & c\bar{z} \end{bmatrix}$ (外積)

  ↓

  張量積 (Tensor product)

⊗ (Kronecker) Tensor product (外積的推廣)

No.
Date:

矩陣

$$A = [a_{ij}]_{m\times n}$$

$$B = [b_{ij}]_{p\times q}$$

$$A \otimes B \overset{\text{def}}{=} \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pq} \end{bmatrix} & \cdots & a_{1n}\begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pq} \end{bmatrix} \\ \vdots & \ddots & \vdots \\ a_{m1}\begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pq} \end{bmatrix} & \cdots & a_{mn}\begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pq} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} & \cdots & a_{11}b_{1q} & & a_{1n}b_{11} & \cdots & a_{1n}b_{1q} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ a_{11}b_{p1} & \cdots & a_{11}b_{pq} & & a_{1n}b_{p1} & \cdots & a_{1n}b_{pq} \\ & \vdots & & \ddots & & \vdots & \\ a_{m1}b_{11} & \cdots & a_{m1}b_{1q} & & a_{mn}b_{11} & \cdots & a_{mn}b_{1q} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ a_{m1}b_{p1} & \cdots & a_{m1}b_{pq} & & a_{mn}b_{p1} & \cdots & a_{mn}b_{pq} \end{bmatrix}_{mp\times nq}$$

向量

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$x \otimes y = \begin{bmatrix} x_1 y \\ x_2 y \\ \vdots \\ x_n y \end{bmatrix} = \begin{bmatrix} x_1 y_1 \\ \vdots \\ x_1 y_m \\ \vdots \\ x_n y_1 \\ \vdots \\ x_n y_m \end{bmatrix} \in \mathbb{C}^{mn}$$

Example

(1)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a * \begin{bmatrix} e & f \\ g & h \end{bmatrix} & b * \begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ c * \begin{bmatrix} e & f \\ g & h \end{bmatrix} & d * \begin{bmatrix} e & f \\ g & h \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{bmatrix}$$

(2)

$$\begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a * \begin{bmatrix} c \\ d \end{bmatrix} \\ b * \begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$$

No.
Date: / /

$$\mathbb{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{matrix} \mathbb{e}_1 \\ \mathbb{e}_2 \end{matrix} \qquad \mathbb{y} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{matrix} \mathbb{e}_1' \\ \mathbb{e}_2' \end{matrix}$$

$$[4 \quad 5]$$

sum (和)

Cartesian product (直積)

$$(\mathbb{e}_1, \mathbb{e}_2') = \mathbb{e}_1 \otimes \mathbb{e}_2'$$

$$\mathbb{x} + \mathbb{y} = \begin{bmatrix} 2+4 \\ 3+5 \end{bmatrix}_2 \qquad \mathbb{x} \times \mathbb{y} = \begin{bmatrix} (2,4) & (2,5) \\ (3,4) & (3,5) \end{bmatrix}_{2\times 2}$$

$$\mathbb{x} \oplus \mathbb{y} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}_{2+2} \qquad \mathbb{x} \otimes \mathbb{y} = \begin{bmatrix} 2\cdot 4 & 2\cdot 5 \\ 3\cdot 4 & 3\cdot 5 \end{bmatrix}_{2\times 2}$$

bilinear

$$f(\mathbb{x}, \mathbb{y}) \to \mathbb{R}$$

direct sum (直和)

row major

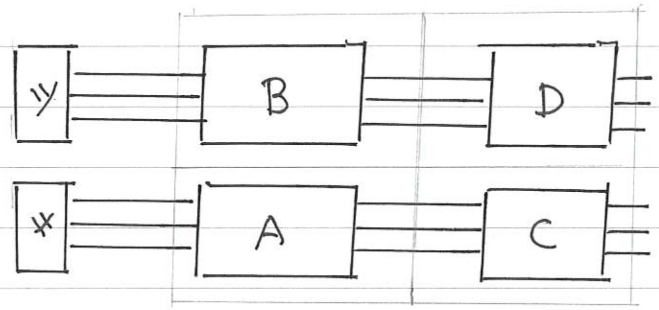
tensor product (張量積)

$$\begin{bmatrix} 2\cdot 4 \\ 2\cdot 5 \\ 3\cdot 4 \\ 3\cdot 5 \end{bmatrix} = \begin{bmatrix} 2\begin{bmatrix} 4 \\ 5 \end{bmatrix} \\ 3\begin{bmatrix} 4 \\ 5 \end{bmatrix} \end{bmatrix}_{2\times 2}$$

linear

$$f(\mathbb{x} \otimes \mathbb{y}) \to \mathbb{R}$$

Quantumn Circuit



$DB\mathbf{y}$

$CA\mathbf{x}$ (串聯)

$\mathbf{x}\otimes\mathbf{y}$ (11)

$A\otimes B$ (並聯)

(10) (11')

$C\otimes D$ (3)

⊗ 性質

(1) $A\otimes(aB+bC)=a(A\otimes B)+b(A\otimes C)$

$(aA+bB)\otimes C=a(A\otimes C)+b(B\otimes C)$

(2) $(A\otimes B)\otimes C=A\otimes(B\otimes C)$

(3) $(A\otimes B)(C\otimes D)=(AC)\otimes(BD)$

(4) (i) $\overline{A\otimes B}=\overline{A}\otimes\overline{B}$

(ii) $(A\otimes B)^T=A^T\otimes B^T$

(iii) $(A\otimes B)^*=A^*\otimes B^*$

(iv) $(A\otimes B)^{-1}=A^{-1}\otimes B^{-1}$

(5) $I_m\otimes I_n=I_{mn}$

(6) $\left\{\begin{matrix}A\\B\end{matrix}\right.$ unitary $\Rightarrow$ (a) $A\otimes B$ (b) AB unitary

(7) $A\otimes B\neq B\otimes A$

(8) $\mathbf{x}\otimes(a\mathbf{y}+b\mathbf{z})=a(\mathbf{x}\otimes\mathbf{y})+b(\mathbf{x}\otimes\mathbf{z})$

$(a\mathbf{x}+b\mathbf{y})\otimes\mathbf{z}=a(\mathbf{x}\otimes\mathbf{z})+b(\mathbf{y}\otimes\mathbf{z})$

(9) $(\mathbf{x}\otimes\mathbf{y})\otimes\mathbf{z}=\mathbf{x}\otimes(\mathbf{y}\otimes\mathbf{z})$

(10) $(A\otimes B)(\mathbf{x}\otimes\mathbf{y})=(A\mathbf{x})\otimes(B\mathbf{y})$

(11) $\|\mathbf{x}\otimes\mathbf{y}\|=\|\mathbf{x}\|\,\|\mathbf{y}\|$

($\|A\mathbf{x}\|=\|\mathbf{x}\|$ A unitary)

証明

(3) $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$

$$\underset{4\times6}{\left(\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\otimes\begin{bmatrix}a&c\\b&d\end{bmatrix}\right)}\underset{6\times4}{\left(\begin{bmatrix}x&u\\y&v\\z&w\end{bmatrix}\otimes\begin{bmatrix}m&p\\n&q\end{bmatrix}\right)} \qquad \underset{2\times2}{\left(\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}x&u\\y&v\\z&w\end{bmatrix}\right)}\otimes\underset{2\times2}{\left(\begin{bmatrix}a&c\\b&d\end{bmatrix}\begin{bmatrix}m&p\\n&q\end{bmatrix}\right)}$$

$$= \begin{bmatrix}1\begin{bmatrix}a&c\\b&d\end{bmatrix} & 2[\quad] & 3[\quad]\\ 4[\quad] & 5[\quad] & 6[\quad]\end{bmatrix}\begin{bmatrix}x\begin{bmatrix}m&p\\n&q\end{bmatrix} & u[\quad]\\ y[\quad] & v[\quad]\\ z[\quad] & w[\quad]\end{bmatrix} \qquad = \begin{bmatrix}1x+2y+3z & 1u+2v+3w\\ 4x+5y+6z & 4u+5v+6w\end{bmatrix}\otimes\left(\begin{bmatrix}a&c\\b&d\end{bmatrix}\begin{bmatrix}m&p\\n&q\end{bmatrix}\right)$$

$$= \begin{bmatrix}1x\begin{bmatrix}a&c\\b&d\end{bmatrix}\begin{bmatrix}m&p\\n&q\end{bmatrix}+2y[\quad][\quad]+3z[\quad][\quad] & 1u[\quad][\quad]+2v[\quad][\quad]+3w[\quad][\quad]\\ 4x[\quad][\quad]+5y[\quad][\quad]+6z[\quad][\quad] & 4u[\quad][\quad]+5v[\quad][\quad]+6w[\quad][\quad]\end{bmatrix}_{4\times4}$$

(2) $(A\otimes B)\otimes C = A\otimes(B\otimes C)$

$$\left(\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\otimes\begin{bmatrix}a&c\\b&d\end{bmatrix}\right)\otimes\begin{bmatrix}m&p\\n&q\end{bmatrix} \qquad \begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\otimes\left(\begin{bmatrix}a&c\\b&d\end{bmatrix}\otimes\begin{bmatrix}m&p\\n&q\end{bmatrix}\right)$$

$$= \begin{bmatrix}1\begin{bmatrix}a&c\\b&d\end{bmatrix} & 2\begin{bmatrix}a&c\\b&d\end{bmatrix} & 3[\quad]\\ 4[\quad] & 5[\quad] & 6[\quad]\end{bmatrix}\otimes\begin{bmatrix}m&p\\n&q\end{bmatrix} \qquad = \begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\otimes\begin{bmatrix}a\begin{bmatrix}m&p\\n&q\end{bmatrix} & c[\quad]\\ b[\quad] & d[\quad]\end{bmatrix}$$

$$= \begin{bmatrix}\begin{bmatrix}1a&1c\\1b&1d\end{bmatrix} & \begin{bmatrix}2a&2c\\2b&2d\end{bmatrix} & \begin{bmatrix}3a&3c\\3b&3d\end{bmatrix}\\ \begin{bmatrix}4a&\\&\end{bmatrix} & \begin{bmatrix}5a&\\&\end{bmatrix} & \begin{bmatrix}6a&\\&\end{bmatrix}\end{bmatrix}\otimes\begin{bmatrix}m&p\\n&q\end{bmatrix}$$

$$= \begin{bmatrix}1\begin{bmatrix}a\begin{bmatrix}m&p\\n&q\end{bmatrix} & c\begin{bmatrix}m&p\\n&q\end{bmatrix}\\ b\begin{bmatrix}m&p\\n&q\end{bmatrix} & d\begin{bmatrix}m&p\\n&q\end{bmatrix}\end{bmatrix} & 2\begin{bmatrix}a[\quad]&c[\quad]\\b[\quad]&d[\quad]\end{bmatrix} & 3\begin{bmatrix}a&c\\b&d\end{bmatrix}\\ 4\begin{bmatrix}a&c\\b&d\end{bmatrix} & 5\begin{bmatrix}a&c\\b&d\end{bmatrix} & 6\begin{bmatrix}a&c\\b&d\end{bmatrix}\end{bmatrix}$$

$$= \begin{bmatrix}\begin{bmatrix}1a\begin{bmatrix}m&p\\n&q\end{bmatrix} & 1c\begin{bmatrix}m&p\\n&q\end{bmatrix}\\ 1b\begin{bmatrix}m&p\\n&q\end{bmatrix} & 1d\begin{bmatrix}m&p\\n&q\end{bmatrix}\end{bmatrix} & \begin{bmatrix}2a[\quad] & 2c[\quad]\\ 2b[\quad] & 2d[\quad]\end{bmatrix} & \begin{bmatrix}3a[\quad] & 3c[\quad]\\ 3b[\quad] & 3d[\quad]\end{bmatrix}\\ \begin{bmatrix}4a & \\ & \end{bmatrix} & \begin{bmatrix}5a & \\ & \end{bmatrix} & \begin{bmatrix}6a & \\ & \end{bmatrix}\end{bmatrix}$$

$A \otimes B$ $\qquad$ $A^T \otimes B^T$

(4)

(iii) $(A\otimes B)^* = \overline{(A\otimes B)^T} = \overline{A^T} \otimes \overline{B^T} = A^* \otimes B^*$

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \otimes \begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\| \qquad\qquad \|$$

(iv) $(A\otimes B)(A^{-1}\otimes B^{-1}) = (AA^{-1}) \otimes (BB^{-1})$

$= I_m \otimes I_n$

$= I_{mn}$

$$\begin{bmatrix} 1\begin{bmatrix} a & c \\ b & d \end{bmatrix} & 4[\ ] \\ 2[\ ] & 5[\ ] \\ 3[\ ] & 6[\ ] \end{bmatrix} \qquad \begin{bmatrix} 1\begin{bmatrix} a & b \\ c & d \end{bmatrix} & 2[\ ] & 3[\ ] \\ 4[\ ] & 5[\ ] & 6[\ ] \end{bmatrix}$$

(5) $I_3 \otimes I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0[\ ] & 0[\ ] \\ 0[\ ] & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0[\ ] \\ 0[\ ] & 0[\ ] & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = I_6$

(6) (a) $\left\{ \begin{array}{l} A \\ B \end{array} \right.$ unitary $\therefore \begin{array}{l} A^*A = I \\ B^*B = I \end{array} \Rightarrow (A\otimes B)^*(A\otimes B) = (A^*\otimes B^*)(A\otimes B)$

$= (A^*A) \otimes (B^*B)$

$= I \otimes I$

$= I$

(b) $(AB)^*(AB) = B^*A^*AB = I$

(11) $x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\qquad \|x\|^2 = a^2+b^2+c^2$ $\qquad$ (假設 $\in \mathbb{R}$)

$y = \begin{bmatrix} u \\ v \end{bmatrix}$ $\qquad \|y\|^2 = u^2+v^2$

$x \otimes y = \begin{bmatrix} au \\ av \\ bu \\ bv \\ cu \\ cv \end{bmatrix}$, $\quad \|x\otimes y\|^2 = a^2u^2+a^2v^2+b^2u^2+b^2v^2+c^2u^2+c^2v^2$

$= (a^2+b^2+c^2)(u^2+v^2)$

$= \|x\|^2 \|y\|^2$

No.
Date: / /

Tensor product of Hilbert Space

- U, V : Hilbert space, orthonormal basis $\mathcal{B}_1 = \{u_1, \cdots, u_m\}$, $\dim U = m$; $\mathcal{B}_2 = \{v_1, \cdots, v_n\}$, $\dim V = n$

$\Rightarrow \mathcal{B} = \{u_i \otimes v_j\}_{\substack{1\le i\le m \\ 1\le j\le n}}$ orthonormal,

$$\because \langle u_i\otimes v_j, u_h \otimes v_k\rangle = (u_i\otimes v_j)^*(u_h\otimes v_k)$$
$$\overset{(4)}{=} (u_i^*\otimes v_j^*)(u_h\otimes v_k)$$
$$\overset{(10)}{=} (u_i^* u_h)\otimes(v_j^* v_k)$$
$$= \delta_{ih}\otimes\delta_{jk}$$
$$= \begin{cases} 1 & i=h \ \& \ j=k \\ 0 & \text{else}\end{cases}$$

- $U\otimes V = \mathrm{span}\left(\{u_i\otimes v_j\}_{\substack{1\le i\le m\\ 1\le j\le n}}\right)$

$$= \left\{\sum_{i,j} a_{ij}\, u_i\otimes v_j \;\middle|\; a_{ij}\in\mathbb{C}\right\}$$

(i) $\dim U\otimes V = mn$

(ii) $(U\otimes V)\otimes W \cong U\otimes (V\otimes W) \overset{\text{def}}{=} U\otimes V\otimes W$

$$U_1\otimes U_2\otimes\cdots\otimes U_k = \bigotimes_{i=1}^{k} U_i$$

Example 1 $\mathcal{H}_2 = \left\{a|0\rangle + b|1\rangle \;\middle|\; {a \atop b}\in\mathbb{C}\right\} = \left\{\begin{bmatrix}a\\b\end{bmatrix} \;\middle|\; {a \atop b}\in\mathbb{C}\right\}$ $\mathcal{B}_1 = \{|0\rangle, |1\rangle\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

- $\mathcal{B} = \{|0\rangle\otimes|0\rangle,\ |0\rangle\otimes|1\rangle,\ |1\rangle\otimes|0\rangle,\ |1\rangle\otimes|1\rangle\}$
$= \{|00\rangle,\ |01\rangle,\ |10\rangle,\ |11\rangle\}$
$= \{|0\rangle,\ |1\rangle,\ |2\rangle,\ |3\rangle\}$
$= \{e_1,\ e_2,\ e_3,\ e_4\}$

$|i\rangle\otimes|j\rangle = |i\rangle|j\rangle = |i,j\rangle = |ij\rangle$

- $\mathcal{H}_2\otimes\mathcal{H}_2 = \mathrm{Span}(\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\})$

$$= \{a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle \mid a_i\in\mathbb{C}\}$$
$$= \{a_0|0\rangle + a_1|1\rangle + a_2|2\rangle + a_3|3\rangle \mid a_i\in\mathbb{C}\}$$
$$= \left\{\begin{bmatrix}a_0\\a_1\\a_2\\a_3\end{bmatrix} \;\middle|\; a_i\in\mathbb{C}\right\} \cong \mathcal{H}_4$$

- $|00\rangle = \begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = e_1$ $|01\rangle = \begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} = e_2$

$|10\rangle = \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} = e_3$ $|11\rangle = \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} = e_4$

$|01\rangle \neq |10\rangle$

$$\mathcal{H}_2 \otimes \mathcal{H}_2 \ni \quad x = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \qquad \text{(entangled)}$$

$$y = \frac{1}{5\sqrt{2}}\left(3|00\rangle + 4|01\rangle + 3|10\rangle + 4|11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \frac{1}{5}\left(3|0\rangle + 4|1\rangle\right) \qquad \text{(factorizable)}$$

Example 2 $\begin{cases} U = \mathcal{H}_4 & \mathcal{B}_1 = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} = \left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} \\ V = \mathcal{H}_2 & \mathcal{B}_2 = \{|0\rangle, |1\rangle\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} \end{cases}$

- $$\mathcal{B} = \left\{\begin{matrix} |00\rangle\otimes|0\rangle, |00\rangle\otimes|1\rangle, |01\rangle\otimes|0\rangle, |01\rangle\otimes|1\rangle, \\ |10\rangle\otimes|0\rangle, |10\rangle\otimes|1\rangle, |11\rangle\otimes|0\rangle, |11\rangle\otimes|1\rangle \end{matrix}\right\}$$
$$= \{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$$
$$= \{|0\rangle, |1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle, |6\rangle, |7\rangle\}$$

- $$\mathcal{H}_4 \otimes \mathcal{H}_2 = \{a_0|0\rangle + a_1|1\rangle + \cdots + a_7|7\rangle \mid a_i \in \mathbb{C}\} \cong \mathcal{H}_8$$

$\vdots$

- $$\bigotimes_{i=1}^{n} \mathcal{H}_2 = \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_2 = \mathcal{H}_2^{\otimes n} \cong \mathcal{H}_{2^n} \qquad \dim = 2^n$$

- $$|000\rangle = |00\rangle\otimes|0\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix} = e_1$$
$$|001\rangle = |00\rangle\otimes|1\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} = e_2$$
$$\vdots \qquad\qquad \vdots$$
$$|101\rangle = |10\rangle\otimes|1\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix} = e_6$$
$$\vdots \qquad\qquad \vdots$$
$$|111\rangle = |11\rangle\otimes|1\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix} = e_8$$

Postulates of Quantumn Mechanics

Bohr → Dirac → Von Neuman

(1) State space Postulate

The state space of a closed quantumn system is described by a a Hilbert space $\mathcal{H}$.

A pure state of the system is described by a unit vector $|\psi\rangle \in \mathcal{H}$

Example A qubit

state space $\mathcal{H} = \{ \alpha |0\rangle + \beta |1\rangle \mid \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1 \}$

$= \mathbb{C}^2 = \{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \mid \text{"} \}$

(2) Evolution Postulate

The time-evolution of the state of a closed quantumn system is described by a unitary operator U. If the initial State is $|\psi_1\rangle$, then after the evolution, the state of the system will be

$$|\psi_2\rangle = U |\psi_1\rangle$$

Example Qubit, state space $\mathcal{H}_2$

Evolution operator $U = H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (Hadamard)

$H|0\rangle = H \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = |+\rangle$

$H|1\rangle = H \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = |-\rangle$

(3) Measurement Postulate

(1) Quantumn system state space $\mathcal{H}$, orthonormal basis $\mathcal{B} = \{|u_1\rangle, \cdots, |u_n\rangle\}$

A Von Neuman measurement on system with state $|\psi\rangle = \sum_i \alpha_i |u_i\rangle \in \mathcal{H}$ outputs i with probability $|\alpha_i|^2$ & leaves the system in state $|u_i\rangle$.

(2) Measurement $= \{M_1, \cdots, M_n\}$, $M_i \in \mathcal{H}^*$, $\sum_i M_i^* M_i = I_n$

After measurement, system in state $|u_i\rangle = \frac{M_i|\psi\rangle}{\sqrt{p_i}}$ with probability $p_i = \| M_i |\psi\rangle \|^2 = \langle \psi M_i^* M_i | \psi \rangle$

Example Qubit, $\mathcal{H}_2$

Measurement $= \{M_1, M_2\}$, $\begin{cases} M_0 = |0\rangle\langle 0| = [1\ 0]\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1&0\\0&0\end{bmatrix} \\ M_1 = |1\rangle\langle 1| = [0\ 1]\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0&0\\0&1\end{bmatrix} \end{cases}$

- $M_i^* = M_i$
- $M_i M_j = \begin{cases} M_i & i=j \\ 0 & i \neq j \end{cases}$

$M_0 + M_1 = \begin{bmatrix}1&0\\0&1\end{bmatrix} = I_2$

- before measurement: in state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

after " : in state $\begin{cases} |0\rangle = \frac{M_0|\psi\rangle}{\sqrt{p_0}}, & p_0 = \| M_0|\psi\rangle \|^2 = |\alpha|^2 \\ |1\rangle = \frac{M_1|\psi\rangle}{\sqrt{p_1}}, & p_1 = \| M_1|\psi\rangle \| = |\beta|^2 \end{cases}$

$$\begin{cases} M_0|\psi\rangle = |0\rangle\langle 0|(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle \\ \quad = \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{bmatrix}\alpha\\0\end{bmatrix} \\ M_1|\psi\rangle = |1\rangle\langle 1|(\alpha|0\rangle + \beta|1\rangle) = \beta|1\rangle \\ \quad = \begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{bmatrix}0\\\beta\end{bmatrix} \end{cases}$$

(4) Composition of System Postulate

(1) Two systems (state space $\mathcal{H}_1, \mathcal{H}_2$) are treated as one combined system.
The state space of the combined system is $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

(2) System 1 in state $|\psi_1\rangle$, then combined system in state $|\psi_1\rangle \otimes |\psi_2\rangle \in \mathcal{H}$
" 2 " $|\psi_2\rangle$ (non-entangled state)

(3) n systems : $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$

Example 2 qubits system

state space $= \mathcal{H}_2 \otimes \mathcal{H}_2 \cong \mathcal{H}_4$

$$\Downarrow$$

state $|\psi\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$

$$\stackrel{?}{=} (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$$

(i) (o) Non-entangled state : $\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$

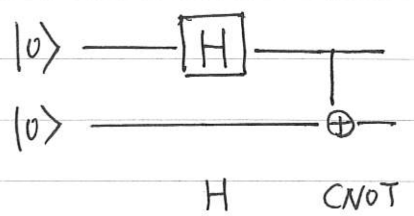
(x) entangled state (糾纏態) : $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (Bell state)

若 $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$

$$= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

$\therefore ad = 0$, $a = 0$ (x)
$d = 0$ (x)

P.S.



$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

No.
Date: / /

- (1) Classical computing $a := b;$ ...
  (2) Cryptography

複製

- No cloning theorem (量子不可克隆定理)

There exists no unitary $U(|u\rangle \otimes |e\rangle) = |u\rangle \otimes |u\rangle$

Pf: otherwise.

$$U\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes |e\rangle\right)$$

$$(1) = \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) = \frac{1}{2}\left(|00\rangle+|01\rangle+|10\rangle+|11\rangle\right)$$

$$(2) = U\left(\frac{|0\rangle}{\sqrt{2}} \otimes |e\rangle + \frac{|1\rangle}{\sqrt{2}} \otimes |e\rangle\right)$$

$$= \frac{1}{\sqrt{2}} U(|0\rangle \otimes |e\rangle) + \frac{1}{\sqrt{2}} U(|1\rangle \otimes |e\rangle)$$

$$= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$ 矛盾